ON A PROPERTY CONCERNING LOCALLY COMPACT GROUPS

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Statement of the result

Let G be a locally compact group. $L^p(G)$ $(1 \leqslant p < \infty)$ is the space of p-th power integrable functions on G, provided with the norm

$$\|f\|_p = (\int |f(x)|^p \, dx)^{1/p}.$$

J. DIEUDONNÉ [1; deuxième partie] has introduced the following property, called (P_p) $(1 \leqslant p < \infty)$, concerning G:

For every compact set $K \subset G$ and every $\varepsilon > 0$ there exists a function $s \in L^p(G)$ such that

(i) $s(x) \geqslant 0$ for all $x \in G$;

(ii) $\|s\|_p = 1$;

(iii) $\|L_y s - s\|_p < \varepsilon$ for all $y \in K$, where L_y is the left translation operator, defined for all functions on G by $(L_y f)(x) = f(y^{-1}x)$.

Dieudonné remarked that if (P_1) holds for a group G, then (P_p) holds for all $p > 1$. H. REITER [2] proved a partial converse, namely that (P_2) implies (P_1). A slight extension of the proofs of these two facts yields a proof that actually the properties (P_1) and (P_p) are equivalent:

Theorem. *Let G be a locally compact group. If (P_{p_0}) holds for G for a fixed p_0 $(1 \leqslant p_0 < \infty)$, then (P_p) holds for G for all p $(1 \leqslant p < \infty)$.*

The property in question can thus be symbolized simply with (P), without any index.

We remark finally that the property (P_∞), defined in an obvious way, is not of interest, because it always trivially holds: one can take $s(x) = 1$ for all $x \in G$.

Proof of the theorem

The proof is based on the following two lemmas.

Lemma 1. *If G has the property (P_p) for a fixed p $(1 \leqslant p < \infty)$, then G has the property (P_r) for all $r \geqslant p$.*

Lemma 2. *If G has the property (P_{2p}) for a fixed p $(1 \leqslant p < \infty)$, then G has the property (P_p).*

Proof of Lemma 1. Let a compact set $K \subset G$ and $\varepsilon > 0$ be given. Let $s_p \in L^p(G)$ satisfy the conditions in the definition of (P_p). Put $s_r = s_p{}^{p/r}$. It is obvious that s_r satisfies the conditions (i) and (ii) of (P_r). For the proof of (iii) we need the well-known elementary inequality $|a-b|^t \leqslant |a^t - b^t|$ $(a \geqslant 0, b \geqslant 0, t \geqslant 1)$, which can be proved by verifying that for $x \geqslant b$ the derivative of the function $f(x) = x^t - b^t - (x-b)^t$ is positive or zero, and that $f(b) = 0$.

For each $y \in K$ we have

$$\|L_y s_r - s_r\|_r = ([\int \{|s_p{}^{p/r}(y^{-1}x) - s_p{}^{p/r}(x)|^{r/p}\}^p \, dx]^{1/p})^{p/r} \leqslant$$
$$\leqslant ([\int |s_p(y^{-1}x) - s_p(x)|^p \, dx]^{1/p})^{p/r} = \|L_y s_p - s_p\|_p{}^{p/r} < \varepsilon^{p/r}.$$

So we only have to start with $\varepsilon^{r/p}$ instead of ε to prove (iii).

Proof of Lemma 2. Let again a compact set $K \subset G$ and $\varepsilon > 0$ be given. Let $s_{2p} \in L^{2p}(G)$ satisfy the conditions in the definition of (P_{2p}). Put $s_p = s_{2p}^2$. It is again obvious that s_p satisfies the conditions (i) and (ii) of (P_p). For the proof of (iii) we need Schwarz's inequality $\|fg\|_1 \leqslant \|f\|_2 \, \|g\|_2$ $(f, g \in L^2(G))$.

For each $y \in K$ we have

$$\|L_y s_p - s_p\|_p = (\int |s_{2p}^2(y^{-1}x) - s_{2p}^2(x)|^p \, dx)^{1/p} \leqslant$$
$$\leqslant (\{\int |s_{2p}(y^{-1}x) + s_{2p}(x)|^{2p} \, dx\}^{\frac{1}{2}} \{\int |s_{2p}(y^{-1}x) - s_{2p}(x)|^{2p} \, dx\}^{\frac{1}{2}})^{1/p} \leqslant$$
$$\leqslant 2 \, \|s_{2p}\|_{2p} \, \|L_y s_{2p} - s_{2p}\|_{2p} < 2\varepsilon.$$

So we have to start with $\frac{1}{2}\varepsilon$ instead of ε to prove (iii).

The proof of the theorem is now immediate: given $p_0 \geqslant 1$, let n be such that $2^n \geqslant p_0$. By Lemma 1, (P_{2^n}) holds; hence, by Lemma 2, applied n times, we obtain (P_1), and then, using Lemma 1 again, we get (P_p) for arbitrary $p \geqslant 1$.

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REFERENCES

1. Dieudonné, J., Sur le produit de composition (II). J. Math. pures et appl., **39**, 275–292 (1960).
2. Reiter, H., Sur la propriété (P_1) et les fonctions de type positif. C.R. Acad. Sc. Paris, t. **258**, p. 5134–5135 (1964).